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Periodic solutions for a class of higher-order Cohen–Grossberg type neural networks with delays

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Abstract

Sufficient conditions are obtained for the existence and global attractivity of periodic solutions of a class of higher-order Cohen–Grossberg type neural networks with delays. The proof is based on Gaines and Mawhin's continuation theorem of coincidence degree theory, the Lyapunov functional and a nonsingular M-matrix. One example is exploited to illustrate the effectiveness of the proposed criteria.

Keywords: Higher-order Cohen–Grossberg neural networks; Periodic solution; Global attractivity; Coincidence degree; Lyapunov functional; Nonsingular M-matrix; Delays

1. Introduction

Over the past few years, the Cohen–Grossberg neural network has been widely studied. The Cohen–Grossberg neural network was introduced by Cohen and Grossberg [1], and is described by the set of ordinary differential equations:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + I_i\right], \quad i = 1, 2, \ldots, n, \tag{1}$$

where $x_i(t)$ is the state variable of the ith neuron; $a_i(\cdot)$ denotes an amplification function; $b_i(\cdot)$ is the behaved function; a_{ij} represents the connection strength from neurons i to j; and $g_j(\cdot)$ represents activation function. The Cohen–Grossberg neural network (1) is a general neural network model, and the Hopfield neural network [2] and the cellular neural network [3] can be considered as its special cases.

In [1], the weight matrix $A \triangleq (a_{ij})_{n\times n}$ is assumed to be symmetric, and the activation function $g_j(\cdot)$ is assumed to be differentiable. Wang et al. established some sufficient conditions for the stability of the system without the constraint of symmetry of the connection matrix and the monotonicity or smoothness of the activation function [4]. In practice, time delays inevitably exist in biological and artificial neural networks due to the finite switching speed of

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neurons and amplifiers, and thus should be incorporated into the model equations. Researchers introduced delays into the system (1) by considering the following system:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij})) + I_i\right], \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\tau_{ij} > 0$ is the transmission delay. This type of networks has been widely studied in recent years, and has found applications in many areas; see for example, [5–10].

For Hopfield, neural networks characterized by first-order interactions are shown to have intrinsic limitations [11–13] ; neural networks with higher-order interactions have gained considerable attention due to the fact that they have a stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks. Sufficient conditions have been successively obtained for the stability of such higher-order neural networks, see [14–19].

As is well known, studies on neural dynamical systems not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillations, bifurcations, and chaos [20–25]. In many applications, the properties of periodic oscillatory solutions are of great interest. For instance, the human brain is in periodic oscillatory or chaos; hence it is of fundamental importance to study periodic oscillatory and chaos phenomena of neural networks. However, to the best of our knowledge, until now few authors have discussed the existence of periodic solutions for the higher-order Cohen–Grossberg neural networks with time delays:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\Bigg[& b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t)) \\ & - \sum_{j=1}^{n} b_{ij}(t) g_j(x_j(t-\tau_j(t))) - \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j(x_j(t-\tau_j(t))) g_l(x_l(t-\tau_l(t))) + I_i(t)\Bigg],\end{aligned} \tag{3}$$

where $i = 1, 2, \ldots, n$; $x_i(t)$ corresponds to the state of the ith unit at time t; $a_{ij}(t)$, $b_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the neural networks, respectively; $I_i(t)$ denotes the external inputs at time t, $\tau_j(t) \ge 0$ are the transmission delays. Throughout this paper, we always assume that $a_{ij}(t)$, $b_{ij}(t)$, $b_{ijl}(t)$, $I_i(t)$ are continuous ω-periodic functions. $\tau_j(t) \ge 0$ are continuously differentiable ω-periodic functions and $0 \le \tau_j'(t) < 1$.

The main objective of this paper is to obtain sufficient conditions for the existence of periodic solutions of (3) by using Gaines and Mawhin's continuation theorem of coincidence degree theory. Moreover, the monotonicity and smoothness of activation functions are not assumed in this paper; nor is the symmetric connection requirement.

The remaining parts of this paper are organized as follows. In the next section (Section 2), we introduce some necessary notations and lemmas which will be used later. In Section 3, sufficient conditions for the existence and global attractivity of periodic solutions of (3) are derived. In Section 4, an illustrative example is given to show the effectiveness of the obtained results. In Section 5, we make some concluding remarks.

2. Preliminaries

In this section, we first introduce some elementary notations, definitions and lemmas which play an important role in the proof of the main results in Section 3.

Let $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ be a matrix, $A > 0$ ($A \ge 0$) denotes each element a_{ij} is positive (nonnegative, respectively); Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be a vector, $x > 0$ ($x \ge 0$) denotes each element x_i is positive (nonnegative, respectively); For $\tau = \max\{\tau_j(t),\ 1 \le j \le n,\ t \in [0, \omega]\}$, $C([-\tau, 0], \mathbb{R}^n)$ denotes the family of continuous functions $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}}$ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \max_{1\le i\le n} \sup_{-\tau\le s\le 0} |\phi_i(s)|$; For every continuous ω-periodic function f, we define

$$[f(t)]^+ = \max_{0\le t\le\omega} |f(t)|, \quad \text{and define} \quad \xi_j = \left(\max_{t\in[0,\omega]} \frac{1}{1-\tau_j'(t)}\right)^{1/2}.$$

To prove our theorems, the properties of a nonsingular M-matrix and some equivalent conditions on checking this nonsingular M-matrix will be used in this paper. To facilitate the reader, we give some results on the nonsingular M-matrix without proof.

Assume that $\mathbb{T}^{n\times n} = \{A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n} \ : \ a_{ij} \le 0,\ i \ne j\}$; then we have

Lemma 1 (*[26]*). *Let $A \in \mathbb{T}^{n\times n}$, then A is a nonsingular M-matrix if and only if one of the following conditions holds:*

(1) *All of the principal minors of A are positive.*
(2) *A is semi-positive; that is, there exists a vector $x > 0$ with $Ax > 0$.*
(3) *A has all positive diagonal elements, and there exists a positive diagonal matrix D such that AD is strictly diagonally dominant; that is:*

$$a_{ii}d_i > \sum_{j\ne i} |a_{ij}|d_j, \quad i = 1, 2, \dots, n.$$

(4) *A is inverse-positive; that is, A^{-1} exists and $A^{-1} \ge 0$.*

Lemma 2 (*[27]*). *Assume that A is a nonsingular M-matrix and*

$$Aw \le d,$$

then

$$w \le A^{-1}d.$$

Let X and Z be two Banach spaces, $L : \mathrm{Dom}\, L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}\, L = \mathrm{codim}\, \mathrm{Im}\, L < +\infty$ *and* $\mathrm{Im}\, L$ *is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projectors*

$$P : X \to X \quad \text{and} \quad Q : Z \to Z$$

such that $\mathrm{Im}\, P = \mathrm{Ker}\, L$, $\mathrm{Ker}\, Q = \mathrm{Im}\, L = \mathrm{Im}(I - Q)$, *it follows that mapping $L|_{\mathrm{Dom}\, L\cap \mathrm{Ker}\, P} : (I - P)X \to \mathrm{Im}\, L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since* $\mathrm{Im}\, Q$ *is isomorphic to* $\mathrm{Ker}\, L$, *there exists an isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$.*

In the sequel, we cite below the continuation theorem [28] as follows.

Lemma 3. *Let X and Z be two Banach spaces and L be a Fredholm mapping of index zero. Assume that $\Omega \subset X$ is an open bounded set and $N : X \to Z$ is a continuous operator which is L-compact on $\bar{\Omega}$. Then $Lx = Nx$ has at least one solution in* $\mathrm{Dom}\, L \cap \bar{\Omega}$, *if the following conditions are satisfied:*

(i) *For each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \mathrm{Dom}\, L$,*

$$Lx \ne \lambda Nx;$$

(ii) *For each $x \in \partial\Omega \cap \mathrm{Ker}\, L$,*

$$QNx \ne 0, \quad \text{and} \quad \deg(JQN, \Omega \cap \mathrm{Ker}\, L, 0) \ne 0,$$

where $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$ is an isomorphism.

Lemma 4 (*[29]*). *Let $f(\cdot) : [0, +\infty) \to \mathbb{R}^+$ be a continuous function. If $f(\cdot)$ is integrable and uniformly continuous on* $[0, +\infty)$, *then* $\lim_{t\to+\infty} f(t) = 0$.

3. Main results

In this section, by using the continuation theorem of Mawhin's coincidence degree theory and the properties of the nonsingular M-matrix, several conditions on the existence and global attractivity of the periodic solution of system (3) are derived.

Theorem 1. *Eq. (3) has at least one ω-periodic solution, if the following conditions are satisfied:*

(a) $a_i(x)$ *is positive and bounded,* $0 < \underline{\alpha}_i \le a_i(x) \le \bar{\alpha}_i$, $i = 1, 2, \ldots, n$;
(b) *There exist* b_i *and* $l_i > 0$ *such that*

$$|b_i(x) - b_i(y)| \le b_i|x - y|, \qquad b_i'(x) \ge l_i > 0, \qquad b_i(0) = 0, \quad \forall x \in \mathbb{R},\ i = 1, 2, \ldots, n;$$

(c) *There exist* G_i, $L_i > 0$ *such that*

$$|g_i(x)| < G_i, \qquad |g_i(x) - g_i(y)| \le L_i|x - y|, \quad \forall x, y \in \mathbb{R}, i = 1, 2, \ldots, n;$$

(d) $E - W$ *is a nonsingular* M*-matrix, where* E *denotes the* $n \times n$ *identity matrix,* $W = (w_{ij})_{n\times n}$ *and*

$$w_{ij} = \frac{\bar{\alpha}_i}{\underline{\alpha}_i l_i}\left([a_{ij}(t)]^+ + [b_{ij}(t)]^+\xi_j + \sum_{l=1}^{n}[b_{ijl}(t)]^+\xi_j G_l\right)L_j, \quad i, j = 1, 2, \ldots, n.$$

Proof. In order to use Lemma 3 for Eq. (3), we shall apply Lemma 3 to construct the set Ω by the method of a priori bounds. We denote by Z (respectively, X) as the set of all continuously (respectively, differentiably) ω-periodic functions $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ defined on $\mathbb{R}$, and write

$$\|x\|_0 = \max_{1\le i\le n}\{[x_i(t)]^+\}, \qquad \|x\|_1 = \max\{\|x\|_0, \|x'\|_0\}.$$

Then X and Z are Banach spaces when they are endowed with the norms $\|\cdot\|_1$ and $\|\cdot\|_0$, respectively.

Let $L : \mathrm{Dom}\ L \subset X \to Z$ and $N : X \to Z$ be given by the following:

$$(Lx)(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x'(t),$$

$$\begin{aligned}(Nx)_i(t) \;=\; -a_i(x_i(t))\Bigg[&b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t-\tau_j(t)))\\ &- \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(x_j(t-\tau_j(t)))g_l(x_l(t-\tau_l(t))) + I_i(t)\Bigg] \triangleq (\bigtriangledown x)_i(t)\end{aligned}$$

where $i = 1, 2, \ldots, n$. It is easy to see that L is a linear operator with

$$\mathrm{Ker}\ L = \mathbb{R}^n, \quad \text{and} \quad \mathrm{Im}\ L = \left\{z \mid z \in Z, \int_0^{\omega} z(t)\mathrm{d}t = 0\right\} \text{ is closed in } Z,$$

and $\dim \mathrm{Ker}\ L = n = \mathrm{codim}\ \mathrm{Im}\ L$, then it follows that L is a Fredholm mapping of index zero.

Denote two continuous projective operators P and Q by

$$Px = \frac{1}{\omega}\int_0^{\omega} x(t)\mathrm{d}t, \quad x \in X; \qquad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\mathrm{d}t, \quad z \in Z.$$

It is not difficult to show that P and Q are continuous projectors satisfying

$$\mathrm{Im}\ P = \mathrm{Ker}\ L \quad \text{and} \quad \mathrm{Im}\ L = \mathrm{Ker}\ Q = \mathrm{Im}(I - Q).$$

Furthermore, the inverse (of $L|_{\mathrm{Dom}\ L\cap\mathrm{Ker}\ P}$) K_P: $\mathrm{Im}\ L \to \mathrm{Ker}\ P \cap \mathrm{Dom}\ L$ exists, which has the form

$$(K_P(z))(t) = \int_0^t z(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\left[\int_0^t z(s)\mathrm{d}s\right]\mathrm{d}t.$$

By some computations, we can show that

$$(QNx)_i(t) = \frac{1}{\omega}\int_0^{\omega}(\bigtriangledown x)_i(t)\mathrm{d}t,$$

$$\begin{aligned}(K_P(I-Q)Nx)_i(t) \;=\; &\int_0^t\left[(\bigtriangledown x)_i(s) - \frac{1}{\omega}\int_0^{\omega}(\bigtriangledown x)_i(\theta)\mathrm{d}\theta\right]\mathrm{d}s\\ &- \frac{1}{\omega}\int_0^{\omega}\int_0^t\left[(\bigtriangledown x)_i(s) - \frac{1}{\omega}\int_0^{\omega}(\bigtriangledown x)_i(\theta)\mathrm{d}\theta\right]\mathrm{d}s\mathrm{d}t,\end{aligned}$$

$$(K_P(I-Q)Nx)'_i(t) = (\nabla x)_i(t) - \frac{1}{\omega}\int_0^{\omega}(\nabla x)_i(\theta)\mathrm{d}\theta.$$

For any bounded open set Ω, there exist some constants $C_i > 0$ such that

$$|(\nabla x)_i(t)| \le C_i,$$

it is easy to see that $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are bounded. Moreover, for any $\epsilon > 0$, there exists $\delta = \epsilon/2C_i$, such that for every $x \in \bar{\Omega}$ implies

$$\begin{aligned}|(K_P(I-Q)Nx)_i(t_1) - (K_P(I-Q)Nx)_i(t_2)| &\le \int_{t_2}^{t_1}\left|(\nabla x)_i(s) - \frac{1}{\omega}\int_0^{\omega}(\nabla x)_i(\theta)\mathrm{d}\theta\right|\mathrm{d}s \\ &\le 2C_i|t_1-t_2| < \epsilon,\end{aligned}$$

for any $0 \le t_1 - t_2 < \delta$ and $t_1,\ t_2 \in [0,\omega]$.

By using the Arzela–Ascoli theorem, for every bounded subset $\Omega \subset X$, $K_P(I-Q)N(\bar{\Omega})$ is relatively compact in X, i.e. N is L-compact on $\bar{\Omega}$.

Consider the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1),$$

i.e.

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\lambda a_i(x_i(t))\Bigg[b_i(x_i(t)) - \sum_{j=1}^n a_{ij}(t)g_j(x_j(t)) - \sum_{j=1}^n b_{ij}(t)g_j(x_j(t-\tau_j(t))) \\ - \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j(x_j(t-\tau_j(t)))g_l(x_l(t-\tau_l(t))) + I_i(t)\Bigg], \quad \lambda \in (0,1).\end{aligned} \tag{4}$$

Assume that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in X$ is a solution of Eq. (4) for a certain $\lambda \in (0,1)$. Multiplying Eq. (4) by $x_i(t)$, and integrating from 0 to ω, we obtain

$$\begin{aligned}0 = -\lambda\int_0^{\omega} x_i(t)a_i(x_i(t))\Bigg[b_i(x_i(t)) - \sum_{j=1}^n a_{ij}(t)g_j(x_j(t)) - \sum_{j=1}^n b_{ij}(t)g_j(x_j(t-\tau_j(t))) \\ - \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j(x_j(t-\tau_j(t)))g_l(x_l(t-\tau_l(t))) + I_i(t)\Bigg],\end{aligned} \tag{5}$$

where $i = 1, 2, \ldots, n$. It follows from (5) that

$$\begin{aligned}\underline{\alpha}_i l_i\int_0^{\omega}|x_i(t)|^2\mathrm{d}t &\le \bar{\alpha}_i\int_0^{\omega}|x_i(t)|\Bigg[\sum_{j=1}^n[a_{ij}(t)]^+\cdot|g_j(x_j(t))| + \sum_{j=1}^n[b_{ij}(t)]^+\cdot|g_j(x_j(t-\tau_j(t)))| \\ &\quad + \sum_{j=1}^n\sum_{l=1}^n[b_{ijl}(t)]^+\cdot|g_j(x_j(t-\tau_j(t)))g_l(x_l(t-\tau_l(t)))| + [I_i(t)]^+\Bigg]\mathrm{d}t \\ &\le \bar{\alpha}_i\int_0^{\omega}|x_i(t)|\Bigg[\sum_{j=1}^n[a_{ij}(t)]^+(L_j|x_j(t)| + |g_j(0)|) \\ &\quad + \sum_{j=1}^n[b_{ij}(t)]^+(L_j|x_j(t-\tau_j(t))| + |g_j(0)|) \\ &\quad + \sum_{j=1}^n\sum_{l=1}^n[b_{ijl}(t)]^+G_l(L_j|x_j(t-\tau_j(t))| + |g_j(0)|) + [I_i(t)]^+\Bigg]\mathrm{d}t \\ &\le \bar{\alpha}_i\sum_{j=1}^n\Bigg[[a_{ij}(t)]^+L_j\int_0^{\omega}|x_i(t)|\cdot|x_j(t)|\mathrm{d}t + [b_{ij}(t)]^+L_j\int_0^{\omega}|x_i(t)|\cdot|x_j(t-\tau_j(t))|\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
& + \sum_{l=1}^{n}[b_{ijl}(t)]^{+} G_l L_j \int_0^{\omega} |x_i(t)| \cdot |x_j(t-\tau_j(t))| \mathrm{d}t \Bigg] \\
& + \bar{\alpha}_i \sum_{j=1}^{n} \left[\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+} G_l \right) |g_j(0)| \right] \int_0^{\omega} |x_i(t)| \mathrm{d}t \\
& + \bar{\alpha}_i [I_i(t)]^{+} \int_0^{\omega} |x_i(t)| \mathrm{d}t \\
\le\ & \bar{\alpha}_i \left\{ [I_i(t)]^{+} + \sum_{j=1}^{n} \left[\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+} G_l \right) |g_j(0)| \right] \right\} \\
& \times \sqrt{\omega} \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \\
& + \bar{\alpha}_i \sum_{j=1}^{n} [a_{ij}(t)]^{+} L_j \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \left(\int_0^{\omega} |x_j(t)|^2 \mathrm{d}t \right)^{1/2} \\
& + \bar{\alpha}_i \sum_{j=1}^{n} [b_{ij}(t)]^{+} L_j \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \left(\int_0^{\omega} |x_j(t-\tau_j(t))|^2 \mathrm{d}t \right)^{1/2} \\
& + \bar{\alpha}_i \sum_{j=1}^{n} \sum_{l=1}^{n} [b_{ijl}(t)]^{+} G_l L_j \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \left(\int_0^{\omega} |x_j(t-\tau_j(t))|^2 \mathrm{d}t \right)^{1/2} .
\end{aligned} \tag{6}
$$

On the other hand, we have

$$
\int_0^{\omega} |x_j(t-\tau_j(t))|^2 \mathrm{d}t \le \xi_j^2 \int_0^{\omega} |x_j(t)|^2 \mathrm{d}t, \quad j = 1, 2, \ldots, n.
$$

This, together with (6), leads to

$$
\begin{aligned}
\underline{\alpha}_i l_i \int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \le\ & \bar{\alpha}_i \sum_{j=1}^{n} \left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} \xi_j + \sum_{l=1}^{n} [b_{ijl}(t)]^{+} \xi_j G_l \right) L_j \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \\
& \times \left(\int_0^{\omega} |x_j(t)|^2 \mathrm{d}t \right)^{1/2} \\
& + \bar{\alpha}_i \left\{ [I_i(t)]^{+} + \sum_{j=1}^{n} \left[\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+} G_l \right) |g_j(0)| \right] \right\} \\
& \times \sqrt{\omega} \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} .
\end{aligned} \tag{7}
$$

For the sake of convenience, we define $\|\phi\|_2$ by

$$
\|\phi\|_2 = \left(\int_0^{\omega} |\phi(t)|^2 \mathrm{d}t \right)^{1/2}, \quad \text{for } \phi \in C(\mathbb{R}, \mathbb{R}).
$$

It follows from (7) that

$$
\begin{aligned}
\|x_i\|_2 \le\ & \frac{\bar{\alpha}_i}{\underline{\alpha}_i l_i} \sum_{j=1}^{n} \left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} \xi_j + \sum_{l=1}^{n} [b_{ijl}(t)]^{+} \xi_j G_l \right) L_j \|x_j\|_2 \\
& + \frac{\bar{\alpha}_i}{\underline{\alpha}_i l_i} \left\{ [I_i(t)]^{+} + \sum_{j=1}^{n} \left[\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+} G_l \right) |g_j(0)| \right] \right\} \sqrt{\omega}
\end{aligned}
$$

$$\triangleq \sum_{j=1}^{n} w_{ij}\|x_j\|_2 + u_i, \tag{8}$$

where

$$w_{ij} = \frac{\bar{\alpha}_i}{\underline{\alpha}_i l_i}\left([a_{ij}(t)]^+ + [b_{ij}(t)]^+\xi_j + \sum_{l=1}^{n}[b_{ijl}(t)]^+\xi_j G_l\right)L_j,$$

$$u_i = \frac{\bar{\alpha}_i}{\underline{\alpha}_i l_i}\left\{[I_i(t)]^+ + \sum_{j=1}^{n}\left[\left([a_{ij}(t)]^+ + [b_{ij}(t)]^+ + \sum_{l=1}^{n}[b_{ijl}(t)]^+ G_l\right)|g_j(0)|\right]\right\}\sqrt{\omega}.$$

Define the vector $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ and $\zeta = (\|x_1\|_2, \|x_2\|_2, \ldots, \|x_n\|_2)^{\mathrm{T}}$; it follows from (8) that

$$(E - W)\zeta \le u,$$

where E denotes the identity matrix, $W = (w_{ij})_{n\times n}$. Since $E - W$ is a nonsingular M-matrix, an application of Lemma 2 yields

$$\zeta \le (E - W)^{-1}u \triangleq v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$$

with $v \ge 0$, which implies that

$$\left(\int_0^{\omega}|x_i(t)|^2\mathrm{d}t\right)^{1/2} \le v_i, \quad i = 1, 2, \ldots, n. \tag{9}$$

It is easy to check that there exists a point $t_i \in [0, \omega]$ such that

$$|x_i(t_i)| \le v_i/\sqrt{\omega} \triangleq R_i, \quad i = 1, 2, \ldots, n. \tag{10}$$

From (10), and since $x_i(t) = x_i(t_i) + \int_{t_i}^{t} x_i'(s)\mathrm{d}s$, it follows that

$$|x_i(t)| \le R_i + \int_0^{\omega}|x_i'(t)|\mathrm{d}t. \tag{11}$$

It follows from system (4) that:

$$\begin{aligned}
\int_0^{\omega}|x_i'(t)|\mathrm{d}t &< \int_0^{\omega}\bar{\alpha}_i b_i|x_i(t)|\mathrm{d}t + \bar{\alpha}_i\sum_{j=1}^{n}\int_0^{\omega}[a_{ij}(t)]^+(L_j|x_j(t)| + |g_j(0)|)\mathrm{d}t \\
&\quad + \bar{\alpha}_i\sum_{j=1}^{n}\int_0^{\omega}[b_{ij}(t)]^+(L_j|x_j(t-\tau_j(t))| + |g_j(0)|)\mathrm{d}t + \bar{\alpha}_i\omega[I_i(t)]^+ \\
&\quad + \bar{\alpha}_i\sum_{j=1}^{n}\sum_{l=1}^{n}\int_0^{\omega}[b_{ijl}(t)]^+ G_l(L_j|x_j(t-\tau_j(t))| + |g_j(0)|)\mathrm{d}t \\
&\le \bar{\alpha}_i b_i\sqrt{\omega}\|x_i\|_2 + \bar{\alpha}_i\sum_{j=1}^{n}\left([a_{ij}(t)]^+ + [b_{ij}(t)]^+\xi_j\right)L_j\sqrt{\omega}\|x_j\|_2 \\
&\quad + \bar{\alpha}_i\sum_{j=1}^{n}\sum_{l=1}^{n}[b_{ijl}(t)]^+\xi_j G_l L_j\sqrt{\omega}\|x_j\|_2 + \bar{\alpha}_i\omega[I_i(t)]^+ \\
&\quad + \bar{\alpha}_i\omega\left[\sum_{j=1}^{n}\left([a_{ij}(t)]^+ + [b_{ij}(t)]^+ + \sum_{l=1}^{n}[b_{ijl}(t)]^+ G_l\right)|g_j(0)|\right] \\
&\le \bar{\alpha}_i b_i\sqrt{\omega}v_i + \bar{\alpha}_i\sum_{j=1}^{n}\left([a_{ij}(t)]^+ + [b_{ij}(t)]^+\xi_j\right)L_j\sqrt{\omega}v_j
\end{aligned}$$

$$
\begin{aligned}
&+\bar{\alpha}_i \sum_{j=1}^{n}\sum_{l=1}^{n}[b_{ijl}(t)]^{+}\xi_j G_l L_j \sqrt{\omega} v_j + \bar{\alpha}_i \omega [I_i(t)]^{+} \\
&+\bar{\alpha}_i \omega \left[\sum_{j=1}^{n}\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l\right)|g_j(0)|\right] \\
\triangleq\ & S_i, \qquad (12)
\end{aligned}
$$

which implies

$$
|x_i(t)| < R_i + S_i. \qquad (13)
$$

In view of (4) and (13), we obtain

$$
\begin{aligned}
|x_i'(t)| &< \bar{\alpha}_i b_i (R_i + S_i) + \bar{\alpha}_i \sum_{j=1}^{n}\left[\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l\right)G_j\right] + \bar{\alpha}_i [I_i(t)]^{+} \\
&\triangleq T_i. \qquad (14)
\end{aligned}
$$

Then, we take

$$
\Omega = \{x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in X;\ \ \max\{[x_i(t)]^{+}, [x_i'(t)]^{+}\} < Z_i\}
$$

with $Z_i = \max\{R_i + S_i, T_i\}$; this satisfies condition (i) in Lemma 3.

When $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \operatorname{Ker} L \cap \partial\Omega = \mathbb{R}^n \cap \partial\Omega$, and x is a constant vector in $\mathbb{R}^n$ with $|x_i| = Z_i$, we have

$$
\begin{aligned}
x_i(QNx)_i &= -x_i a_i(x_i) b_i(x_i) + x_i a_i(x_i) \sum_{j=1}^{n}\left[g_j(x_j)\frac{1}{\omega}\int_0^{\omega} a_{ij}(t)\mathrm{d}t + g_j(x_j)\frac{1}{\omega}\int_0^{\omega} b_{ij}(t)\mathrm{d}t\right] \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} g_j(x_j) g_l(x_l)\frac{1}{\omega}\int_0^{\omega} b_{ijl}(t)\mathrm{d}t - \frac{x_i a_i(x_i)}{\omega}\int_0^{\omega} I_i(t)\mathrm{d}t \\
&\le -l_i \underline{\alpha}_i x_i^2 + |x_i|\bar{\alpha}_i \sum_{j=1}^{n}\left[([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+})(L_j|x_j| + |g_j(0)|)\right] \\
&\quad + |x_i|\bar{\alpha}_i \sum_{j=1}^{n}\sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l(L_j|x_j| + |g_j(0)|) + |x_i|\bar{\alpha}_i [I_i(t)]^{+} \\
&= -l_i \underline{\alpha}_i Z_i^2 + Z_i \bar{\alpha}_i \sum_{j=1}^{n}([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+}) L_j Z_j + Z_i \bar{\alpha}_i \sum_{j=1}^{n}\sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l L_j Z_j \\
&\quad + Z_i \bar{\alpha}_i \sum_{j=1}^{n}\left[\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l\right)|g_j(0)|\right] + Z_i \bar{\alpha}_i [I_i(t)]^{+} \\
&= l_i Z_i \underline{\alpha}_i \left\{-Z_i + \frac{\bar{\alpha}_i}{l_i \underline{\alpha}_i}\sum_{j=1}^{n}\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l\right)L_j Z_j\right. \\
&\quad \left. + \frac{\bar{\alpha}_i}{l_i \underline{\alpha}_i}\left[[I_i(t)]^{+} + \sum_{j=1}^{n}\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l\right)|g_j(0)|\right]\right\} \\
&\le l_i Z_i \underline{\alpha}_i \left\{-Z_i + \frac{\bar{\alpha}_i}{l_i \underline{\alpha}_i}\sum_{j=1}^{n}\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+}\xi_j + \sum_{l=1}^{n}[b_{ijl}(t)]^{+}\xi_j G_l\right)L_j Z_j\right. \\
&\quad \left. + \frac{\bar{\alpha}_i}{l_i \underline{\alpha}_i}\left[[I_i(t)]^{+} + \sum_{j=1}^{n}\left([a_{ij}(t)]^{+} + [b_{ij}(t)]^{+} + \sum_{l=1}^{n}[b_{ijl}(t)]^{+}G_l\right)|g_j(0)|\right]\right\}
\end{aligned}
$$

$$= l_i Z_i \underline{\alpha}_i \left(-Z_i + \sum_{j=1}^{n} w_{ij} Z_j + \frac{u_i}{\sqrt{\omega}} \right),$$

since $Z_i > R_i = v_i/\sqrt{\omega}$; that is

$$Z > R = \frac{v}{\sqrt{\omega}} = (E - W)^{-1} \frac{u}{\sqrt{\omega}},$$

where $Z = (Z_1, Z_2, \ldots, Z_n)^{\mathrm{T}}$, $R = (R_1, R_2, \ldots, R_n)^{\mathrm{T}}$, $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$, i.e. $(E - W)Z \neq u/\sqrt{\omega}$. A direct calculation shows that there exists some i such that

$$-Z_i + \sum_{j=1}^{n} w_{ij} Z_j + \frac{u_i}{\sqrt{\omega}} < 0.$$

This leads to

$$x_i(QNx)_i < 0, \quad \text{and} \quad \|QNx\|_0 > 0 \tag{15}$$

which implies $QNx \neq 0$. Let $h(x, \theta) = -\theta x + (1 - \theta)QNx$, $\theta \in [0, 1]$. When $x \in \operatorname{Ker} L \cap \partial\Omega$, it follows from (15) that $\|h(x, \theta)\|_0 > 0$, that is $h(x, \theta) \neq 0$. According to the invariance of the homology, we obtain

$$\deg\{JQNx, \Omega \cap \operatorname{Ker} L,\ 0\} = \deg\{-x, \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0$$

where J: Im $Q \to$ Ker L is an isomorphism. By now we have proved that Ω satisfies all the requirements in Lemma 3. Therefore, Eq. (3) has at least one ω-periodic solution, and this completes the proof. □

Theorem 2. *Eq. (3) has a unique ω-periodic solution, which is globally attractive, if the following conditions hold:*

(a) *$a_i(x)$ is positive, bounded, and there exist $\underline{\alpha}_i, \bar{\alpha}_i, a_i > 0$ such that*

$$0 < \underline{\alpha}_i \le a_i(x) \le \bar{\alpha}_i, \qquad |a_i(x) - a_i(y)| \le a_i|x - y|, \quad \forall x, y \in \mathbb{R},\ i = 1, 2, \ldots, n;$$

(b) *There exist b_i and $l_i > 0$ such that*

$$|b_i(x) - b_i(y)| \le b_i|x - y|, \qquad b_i'(x) \ge l_i > 0, \qquad b_i(0) = 0, \quad \forall x \in \mathbb{R},\ i = 1, 2, \ldots, n;$$

(c) *There exist G_i, $L_i > 0$ such that*

$$|g_i(x)| < G_i, \qquad |g_i(x) - g_i(y)| \le L_i|x - y|, \quad \forall x, y \in \mathbb{R},\ i = 1, 2, \ldots, n;$$

(d) *$E - W$ is a nonsingular M-matrix, where $W = (w_{ij})_{n\times n}$ and*

$$w_{ij} = \frac{\bar{\alpha}_i}{\underline{\alpha}_i l_i} \left([a_{ij}(t)]^+ + [b_{ij}(t)]^+ \xi_j + \sum_{l=1}^{n} [b_{ijl}(t)]^+ \xi_j G_l \right) L_j, \quad i, j = 1, 2, \ldots, n;$$

(e) *$\Gamma = -(K + \Sigma)$ is a nonsingular M-matrix, where $K = \operatorname{diag}\{k_1, k_2, \ldots, k_n\}$, $\Sigma = (\sigma_{ij})_{n\times n}$ and*

$$k_i = -\underline{\alpha}_i l_i + \frac{a_i C_i}{\underline{\alpha}_i} + \frac{\bar{\alpha}_i}{2} \sum_{j=1}^{n} \left[\left([a_{ij}(t)]^+ + [b_{ij}(t)]^+ + \sum_{l=1}^{n} ([b_{ijl}(t)]^+ + [b_{ilj}(t)]^+) G_l \right) L_j \right],$$

$$\sigma_{ij} = \frac{\bar{\alpha}_i}{2} \left\{ [a_{ij}(t)]^+ + \frac{[b_{ij}(t)]^+}{1 - [\tau_j'(t)]^+} + \sum_{l=1}^{n} \frac{[b_{ijl}(t)]^+ + [b_{ilj}(t)]^+}{1 - [\tau_j'(t)]^+} G_l \right\} L_j.$$

Proof. By Theorem 1, Eq. (3) has at least one ω-periodic solution $y(t)$ under the stated conditions. Let $x(t)$ be any solution of Eq. (3) with the initial function $\varphi \in C([-\tau, 0], \mathbb{R}^n)$. Denote $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))^{\mathrm{T}}$ with $\gamma_i(t) = x_i(t) - y_i(t)$; $f_j(\gamma_j(t)) = g_j(\gamma_j(t) + y_j(t)) - g_j(y_j(t))$, and then $\gamma_i(t)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}\gamma_i(t)}{\mathrm{d}t} = -a_i(x_i(t)) \Bigg\{ & b_i(x_i(t)) - b_i(y_i(t)) - \sum_{j=1}^{n} a_{ij}(t) f_j(\gamma_j(t)) - \sum_{j=1}^{n} b_{ij}(t) f_j(\gamma_j(t - \tau_j(t))) \\ & - \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl}(t)[f_j(\gamma_j(t - \tau_j(t))) g_l(x_l(t - \tau_l(t))) + f_l(\gamma_l(t - \tau_l(t))) g_j(y_j(t - \tau_j(t)))] \Bigg\}\end{aligned}$$

$$
\begin{aligned}
&-[a_i(x_i(t))-a_i(y_i(t))]\left[b_i(y_i(t))-\sum_{j=1}^{n}a_{ij}(t)g_j(y_j(t))-\sum_{j=1}^{n}b_{ij}(t)g_j(y_j(t-\tau_j(t)))\right.\\
&\left.-\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)g_j(y_j(t-\tau_j(t)))g_l(y_l(t-\tau_l(t)))+I_i(t)\right]\\
=&-a_i(x_i(t))\left\{b_i(x_i(t))-b_i(y_i(t))-\sum_{j=1}^{n}a_{ij}(t)f_j(\gamma_j(t))-\sum_{j=1}^{n}b_{ij}(t)f_j(\gamma_j(t-\tau_j(t)))\right.\\
&\left.-\sum_{j=1}^{n}\sum_{l=1}^{n}[b_{ijl}(t)g_l(x_l(t-\tau_l(t)))+b_{ilj}(t)g_l(y_l(t-\tau_l(t)))]f_j(\gamma_j(t-\tau_j(t)))\right\}\\
&-[a_i(x_i(t))-a_i(y_i(t))]\left[b_i(y_i(t))-\sum_{j=1}^{n}a_{ij}(t)g_j(y_j(t))-\sum_{j=1}^{n}b_{ij}(t)g_j(y_j(t-\tau_j(t)))\right.\\
&\left.-\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)g_j(y_j(t-\tau_j(t)))g_l(y_l(t-\tau_l(t)))+I_i(t)\right].
\end{aligned}
$$

Hence, we can easily obtain the following differential inequality:

$$
\begin{aligned}
|\gamma_i(t)|D^+|\gamma_i(t)| \le\ & -\underline{\alpha}_i l_i|\gamma_i(t)|^2+\bar{\alpha}_i\sum_{j=1}^{n}[a_{ij}(t)]^+L_j|\gamma_j(t)|\cdot|\gamma_i(t)|\\
&+\bar{\alpha}_i\sum_{j=1}^{n}[b_{ij}(t)]^+L_j|\gamma_j(t-\tau_j(t))|\cdot|\gamma_i(t)|\\
&+\bar{\alpha}_i\sum_{j=1}^{n}\sum_{l=1}^{n}([b_{ijl}(t)]^++[b_{ilj}(t)]^+)G_lL_j|\gamma_j(t-\tau_j(t))|\cdot|\gamma_i(t)|+a_i|\gamma_i(t)|^2C_i/\underline{\alpha}_i\\
\le\ & -\underline{\alpha}_i l_i|\gamma_i(t)|^2+\bar{\alpha}_i\sum_{j=1}^{n}[a_{ij}(t)]^+L_j\frac{|\gamma_j(t)|^2+|\gamma_i(t)|^2}{2}\\
&+\bar{\alpha}_i\sum_{j=1}^{n}[b_{ij}(t)]^+L_j\frac{|\gamma_j(t-\tau_j(t))|^2+|\gamma_i(t)|^2}{2}\\
&+\bar{\alpha}_i\sum_{j=1}^{n}\sum_{l=1}^{n}([b_{ijl}(t)]^++[b_{ilj}(t)]^+)G_lL_j\frac{|\gamma_j(t-\tau_j(t))|^2+|\gamma_i(t)|^2}{2}\\
&+a_i|\gamma_i(t)|^2C_i/\underline{\alpha}_i
\end{aligned}
\tag{16}
$$

where C_i is the bound of $|(\triangledown y)_i(t)|$. We define the following Lyapunov functionals:

$$
\begin{aligned}
V_i(t)=\ &\frac{|\gamma_i(t)|^2}{2}+\frac{\bar{\alpha}_i}{2}\sum_{j=1}^{n}[b_{ij}(t)]^+L_j\frac{1}{1-[\tau_j'(t)]^+}\int_{t-\tau_j(t)}^{t}|\gamma_j(s)|^2\mathrm{d}s\\
&+\frac{\bar{\alpha}_i}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}([b_{ijl}(t)]^++[b_{ilj}(t)]^+)G_lL_j\frac{1}{1-[\tau_j'(t)]^+}\int_{t-\tau_j(t)}^{t}|\gamma_j(s)|^2\mathrm{d}s,
\end{aligned}
\tag{17}
$$

where $i=1,2,\dots,n$.

Calculating and estimating the time-derivative of $V_i(t)$ and noting that $1-\tau_j'(t)\ge 1-[\tau_j'(t)]^+$, one obtains

$$
V_i'(t)\le\left\{-\underline{\alpha}_i l_i+\frac{a_iC_i}{\underline{\alpha}_i}+\frac{\bar{\alpha}_i}{2}\sum_{j=1}^{n}\left[\left([a_{ij}(t)]^++[b_{ij}(t)]^++\sum_{l=1}^{n}([b_{ijl}(t)]^++[b_{ilj}(t)]^+)G_l\right)L_j\right]\right\}|\gamma_i(t)|^2
$$

$$+\frac{\bar{\alpha}_i}{2}\sum_{j=1}^{n}\left\{[a_{ij}(t)]^{+}+[b_{ij}(t)]^{+}\frac{1}{1-[\tau_j'(t)]^{+}}+\sum_{l=1}^{n}([b_{ijl}(t)]^{+}\right.$$
$$\left.+[b_{ilj}(t)]^{+})\frac{1}{1-[\tau_j'(t)]^{+}}G_l\right\}L_j|\gamma_j(t)|^2$$
$$\triangleq k_i|\gamma_i(t)|^2+\sum_{j=1}^{n}\sigma_{ij}|\gamma_j(t)|^2 \tag{18}$$

where

$$k_i=-\underline{\alpha}_i l_i+\frac{a_iC_i}{\underline{\alpha}_i}+\frac{\bar{\alpha}_i}{2}\sum_{j=1}^{n}\left[\left([a_{ij}(t)]^{+}+[b_{ij}(t)]^{+}+\sum_{l=1}^{n}([b_{ijl}(t)]^{+}+[b_{ilj}(t)]^{+})G_l\right)L_j\right],$$
$$\sigma_{ij}=\frac{\bar{\alpha}_i}{2}\left\{[a_{ij}(t)]^{+}+[b_{ij}(t)]^{+}\frac{1}{1-[\tau_j'(t)]^{+}}+\sum_{l=1}^{n}([b_{ijl}(t)]^{+}+[b_{ilj}(t)]^{+})\frac{1}{1-[\tau_j'(t)]^{+}}G_l\right\}L_j.$$

Let $V(t)=(V_1(t),V_2(t),\dots,V_n(t))^{\mathrm{T}}$; $K=\operatorname{diag}\{k_1,k_2,\dots,k_n\}$; $\Sigma=(\sigma_{ij})_{n\times n}$; $w=(|\gamma_1(t)|^2,|\gamma_2(t)|^2,\dots,|\gamma_n(t)|^2)^{\mathrm{T}}$; then (18) can be rewritten in the following vector-matrix form

$$V'(t)\le(K+\Sigma)w.$$

Since $\Gamma=-(K+\Sigma)$ is a nonsingular M-matrix, it follows from Lemma 2 that

$$\Gamma^{-1}V'(t)\le -w.$$

Define the vector $\bar{V}(t)=(\bar{V}_1(t),\bar{V}_2(t),\dots,\bar{V}_n(t))^{\mathrm{T}}=\Gamma^{-1}V(t)\ge 0$; it follows that

$$\bar{V}_i'(t)\le -|\gamma_i(t)|^2. \tag{19}$$

Integrating both sides of (19) from 0 to t results in

$$\bar{V}_i(t)+\int_0^t|\gamma_i(s)|^2\mathrm{d}s\le\bar{V}_i(0)=(\Gamma^{-1}V(0))_i<+\infty,\quad t\ge 0, \tag{20}$$

and hence,

$$\int_0^t|\gamma_i(s)|^2\mathrm{d}s\le\bar{V}_i(0)<+\infty,$$

which implies that $|\gamma_i(t)|^2$ is integrable on $[0,+\infty)$, $i=1,2,\dots,n$. By some computations, it follows from (17) and (20) that

$$\Gamma^{-1}(|\gamma_1(t)|^2,|\gamma_2(t)|^2,\dots,|\gamma_n(t)|^2)^{\mathrm{T}}\le 2\bar{V}(t)\le 2\bar{V}(0)<+\infty,\quad t>0,$$

which means $|\gamma_i(t)|<+\infty$, i.e. $|x_i(t)-y_i(t)|<+\infty$. Therefore, $y_i(t)$ being bounded implies that $x_i(t)$ is bounded; this, together with (3), leads to the boundedness of $x_i'(t)$ and $y_i'(t)$; then $|\gamma_i(t)|^2$ is uniformly continuous on $[0,+\infty)$. An application of Lemma 4 yields

$$\lim_{t\to+\infty}|\gamma_i(t)|^2=0,$$

that is,

$$\lim_{t\to+\infty}|x_i(t)-y_i(t)|^2=0,$$

and this completes the proof. □

Remark. Since there are many equivalent conditions for checking a nonsingular M-matrix given by Berman and Plemmons [26], our theorems obtained here can be easily verified. In addition, the theoretical results obtained here can be used to study some other nonlinear dynamical systems, for example, the Hopfield neural networks, cellular neural networks, etc. Therefore, the results obtained in this paper are significant in both theory and application.

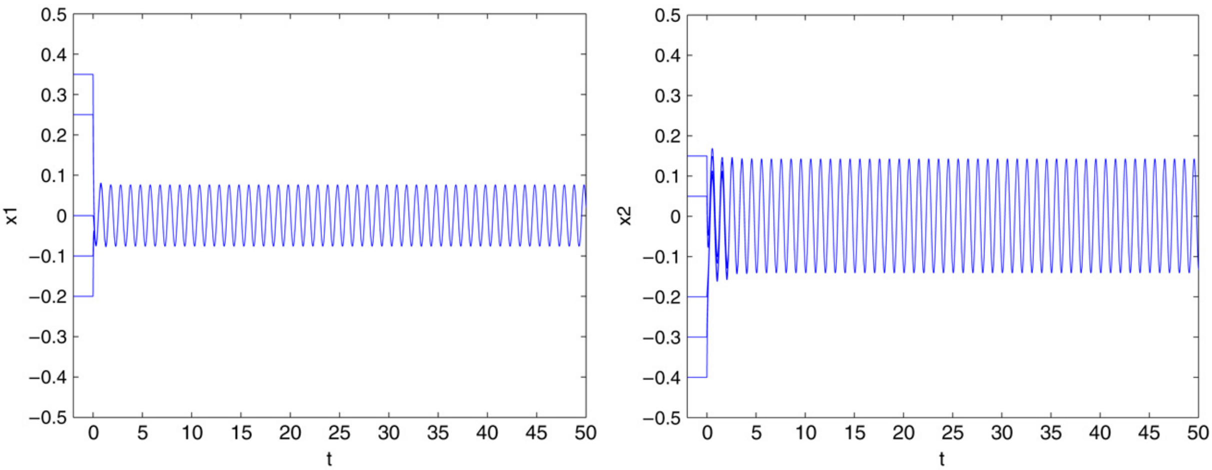

Fig. 1. Transient responses of state variables $x_1(t)$ and $x_2(t)$.

4. An illustrative example

Example. Consider the following higher-order Cohen–Grossberg type neural networks with time delays:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{2} a_{ij} g_j(x_j(t)) - \sum_{j=1}^{2} b_{ij} g_j(x_j(t-\tau_j)) - \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl} g_j(x_j(t-\tau_j)) g_l(x_l(t-\tau_l)) + I_i(t)\right] \tag{21}$$

where $g_j(x) \equiv (1/2)(|x+1| - |x-1|)$; $a_1(x) = (1/50)\sin x + 5$, $a_2(x) = (1/50)\cos x + 7$; $b_1(x) = 4x$, $b_2(x) = 2x$; $I_1(t) = \sin(2\pi t)$, $I_2(t) = \cos(2\pi t)$; $\tau_1 = 1$, $\tau_2 = 2$;

$$(a_{ij})_{2\times 2} = \begin{bmatrix} 0.02 & -0.1 \\ -0.2 & 0.03 \end{bmatrix}, \qquad (b_{ij})_{2\times 2} = \begin{bmatrix} 0.03; & 0 \\ 0.1 & 0.2; \end{bmatrix},$$

$$(b_{1jl})_{2\times 2} = \begin{bmatrix} 0.02; & 0.3 \\ -0.1; & 0 \end{bmatrix}, \qquad (b_{2jl})_{2\times 2} = \begin{bmatrix} -0.03 & 0 \\ 0.2 & 0.14 \end{bmatrix}.$$

By some computation, we have

$$E - W = \begin{bmatrix} 0.9068 & -0.0504 \\ -0.1659 & 0.7134 \end{bmatrix}, \qquad C_1 = 272.3873, \qquad C_2 = 330.7951;$$

$$\Gamma = \begin{bmatrix} 15.1113 & -1.2550 \\ -1.9656 & 6.0624 \end{bmatrix}.$$

It is easy to check that $E - W$ and Γ are nonsingular M-Matrices by Lemma 1. We easily check that the conditions in Theorem 2 hold, and hence the model (21) has a unique 1-periodic solution, which is globally attractive. Fig. 1 depicts the time responses of state variables $x_1(t)$, $x_2(t)$ with five different initial values, respectively, and the step $h = 0.01$. Fig. 2 depicts the phase plots of state variables $x_1(t)$, $x_2(t)$. It confirms that the proposed conditions in Theorem 2 lead to the unique and globally attractive 1-periodic solution for Eq. (21).

5. Conclusions

In this paper, we have studied the existence of periodic solutions for a class of higher-order Cohen–Grossberg type neural networks with time delays. Sufficient conditions have been derived to ascertain the existence and the global attractivity of the periodic solution without the constraints of symmetry of the connection matrix, monotonicity, and

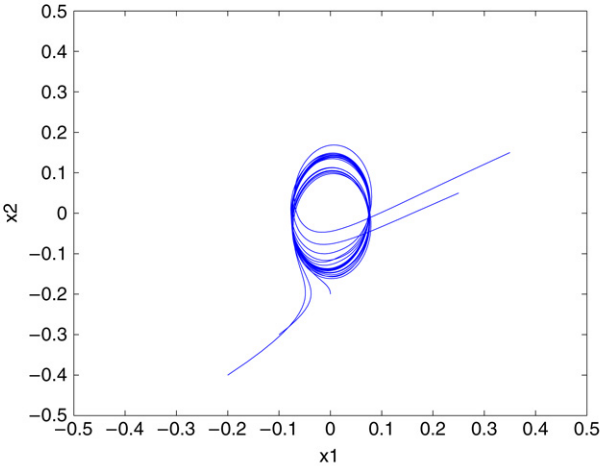

Fig. 2. Phase plots of state variables $x_1(t)$ and $x_2(t)$.

smoothness of the activation function. The proposed model has generalized the classical Cohen–Grossberg model, as well as Hopfield neural networks. In addition, since there are many equivalent conditions for checking a nonsingular M-matrix given by Berman and Plemmons [26], our theorems obtained here can be easily verified in practice.

The parameter uncertainties and stochastic noises are also important topics in neural networks [10,30,31]. As a future topic, the parameter uncertainties and stochastic noises from a variety of sources will be incorporated in the higher-order neural network model. In addition, higher-order neural networks with mixed (both discrete and distributed) features will be discussed in our next paper.

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